Preliminaries of Statistical Depth

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Introduction to Statistical Depth Function and Tukey's Halfspace Depth

Statistical depth functions [4] provide a way to define the "depth" or centrality of a point with respect to a probability distribution in multivariate settings. A depth function, denoted as $D(x; P)$, assigns a depth value to each point $x \in \mathbb{R}^d$ with respect to a distribution P. This concept helps to generalize univariate rank and order statistics into higher dimensions, allowing us to understand the relative centrality of data points within a distribution.

One of the foundational depth functions is Tukey's "halfspace" depth, introduced by Tukey in 1975. The halfspace depth of a point x in $\mathbb{R}^d$ is defined as the minimum probability mass contained within any closed halfspace that includes x. Mathematically, it is given by:

$$HD(x; P) = \inf\{P(H) : H \text{ is a closed halfspace}, x \in H\}$$

Tukey's contribution [3] is to generalize the concept of ranking to 2d. Tukey suggested treating order statistics as oriented points that could define lines with certain properties in the plane. Specifically, in the plane, an (i, j) line would contain $\geq i$ points to its right (or on it) and $\leq j$ points strictly to its left. For any integer $i < n$, there exists a unique line of depth i that forms a closed curve, called the i-loop, enclosing a specific depth contour.

Axioms of Depth Function

Liu, Zuo, Serfling [4] unify the depth function by four axioms.

(A1) (Affine invariance) $D_{P_{AX+b}}(Ax + b) = D_{P_X}(x)$ for any $x \in \mathbb{R}^d$, any full-rank $d \times d$ matrix A, and any $b \in \mathbb{R}^d$.

(A2) (Maximality at the center) If x_0 is a center of symmetry for P (symmetry here can be either *central*, *angular* or *halfspace* symmetry), it is *deepest*, that is, $D_P(x_0) = \max_{x\in\mathbb{R}^d} D_P(x)$.

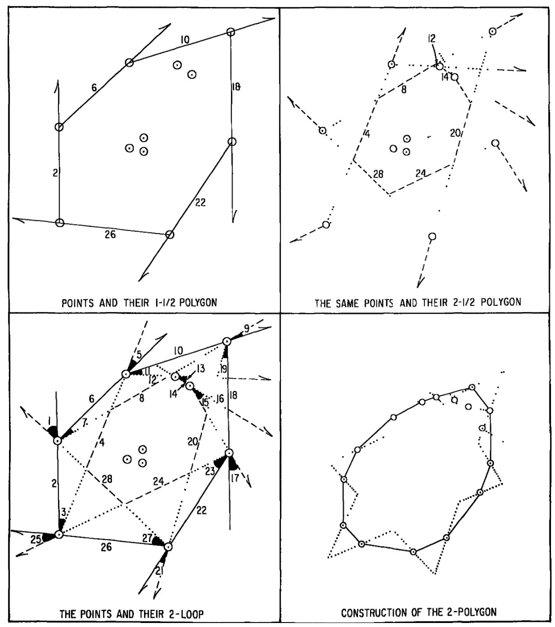

Figure 1: Illustration of depth contours: the 1-loop, $1\frac{1}{2}$-polygon, and other depth-based polygons in two dimensions.

(A3) (Linear monotonicity relative to the deepest points) If $D_P(x_0) = \max_{x \in \mathbb{R}^d} D_P(x)$, then $D_P(x) \leq D_P((1-\alpha)x_0 + \alpha x)$ for all $\alpha \in [0, 1]$ and $x \in \mathbb{R}^d$; depth is monotonically decreasing along any straight line running through a deepest point.

(A4) (Vanishing at infinity) $\lim_{\|x\| \to \infty} D_P(x) = 0$.

Chernozhukov [1] proposes below a new depth concept, the Monge-Kantorovich (MK) depth, that relinquishes the affine equivariance and star convexity of contours imposed by Axioms (A1) and (A3) and recovers non convex features of the underlying distribution.

MK Depth

In Chernozhukov's paper [1], he takes a totally different and more agnostic approach, on the model of the discussion by Serfling [2]: if the ultimate purpose of statistical depth is to provide, for each distribution P, a P-related ordering of $\mathbb{R}^d$ producing adequate concepts of quantile and distribution functions, ranks and signs, the relevance of a given depth function should be evaluated in terms of the relevance of the resulting ordering, and the quantiles, ranks and signs it produces.

References

[1] Victor Chernozhukov, Alfred Galichon, Marc Hallin, and Marc Henry. Monge–kantorovich depth, quantiles, ranks and signs. 2017.

[2] Robert Serfling. Quantile functions for multivariate analysis: approaches and applications. *Statistica Neerlandica*, 56(2):214–232, 2002.

[3] John W Tukey. Mathematics and the picturing of data. In *Proceedings of the international congress of mathematicians*, volume 2, pages 523–531. Vancouver, 1975.

[4] Yijun Zuo and Robert Serfling. General notions of statistical depth function. *Annals of statistics*, pages 461–482, 2000.