M be a cpt d-dim smooth mfd embedded in $\mathbb{R}^D$.

($\exists$ smooth injective immersion from M to $\mathbb{R}^D$, that is homeomorphic onto its image.

$S^2 = \{ \vec{z} \in \mathbb{R}^3 : x^2+y^2+z^2=1 \}$

Tangent space. (Refer to diff geom. 2.4). Def^{n} 2.2.21.

Gradient.

$\gamma : [a,b] \to M$. $\quad l(\gamma) = \int_a^b |\gamma'(t)| dt$.

arclength. $s(t) = \int_a^t |\gamma'(u)| du$.

$s'(t) = |\gamma'(t)|$.

$t = t(s)$ be the inverse function of $s(t)$. $\quad J = s([a,b])$.

Then $\beta(s) = \gamma(t(s))$.

$\beta'(s) = \gamma'(t)\, t'(s)$.

$|\beta'(s)| = |\gamma'(t)|\, |t'(s)| = |s'(t)| \frac{1}{|s'(t)|} = 1$.

$\beta(u) = \alpha(t(u))$.

$\int_{t_0}^{t} |\alpha'(t_1)| dt_1 = \int_{u_0}^{u} |\beta'(u_1)| du$.

$\gamma: [a,b] \to M$. $\quad s(t) = \int_a^t |\gamma'(u)|\, du \quad s'(t) = |\gamma'(t)|$

$\int_0^{s(b)} |\beta'(s)|\, ds \qquad 0 \le s \le s(b)$

$= \cancel{\int_0^{s(b)} |\beta'(t) t'(s)|\, ds} \qquad \beta(s) = \gamma \circ t(s) \qquad \int_a^b |\gamma'(u)|\, du$

$= \int_0^{s(b)} |\gamma'(t)\, t'(s)|\, ds \qquad = s(b)$

$= \int_0^{s(b)} |s'(t)|\, |t'(s)|\, ds$

$= s(b)$

$\tilde{s}(t) = \dfrac{\int_a^t |\gamma'(u)|\, du}{s(b)} \qquad 0 \le \tilde{s} \le 1. \qquad \tilde{s}'(t) = \dfrac{|\gamma'(t)|}{s(b)}$

$\int_0^1 |\beta'(\tilde{s})|\, d\tilde{s} \qquad |\gamma'(t)| = \tilde{s}'(t)\, s(b)$

$= \int_0^1 |\gamma'(t)|\, |t'(\tilde{s})|\, d\tilde{s}$

$= \int_0^1 \tilde{s}'(t)\, s(b)\, |t'(\tilde{s})|\, d\tilde{s}$

$= s(b)$.

So use the above parametrization.

$\int_a^b |\gamma'(t)|\, dt$ inf can be taken to be over curves of constant speed on the unit interval. (like above)

$\Rightarrow$ We have $\left(\int_0^1 |\gamma'(t)|\, dt\right)^2 = \int_0^1 |\gamma'(t)|^2\, dt$.

First $\ge$ second.

$\forall\, \gamma: [0,1] \to M \quad \gamma(0) = x \quad \gamma(1) = y$.

$\int_0^1 |\gamma'(t)|^2\, dt \ge \left(\int_0^1 |\gamma'(t)|\, dt\right)^2$.

Second $\ge$ First.

$\Rightarrow$ We have equivalent definitions.

$\gamma'(s) \cdot \gamma'(s) = 1$ (parametrize by arc length)

$\gamma''(s) \cdot \gamma'(s) = 0$.

$\Rightarrow \gamma''(s) \perp T_{\gamma(s)} M$.

$\frac{d}{dt} |\dot{\gamma}|^2 = 2\langle \dot{\gamma}, \ddot{\gamma} \rangle = 0 \quad \Rightarrow$ geodsesic has constant speed.

$\mathrm{div}(v) = \nabla \cdot v = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_d}{\partial x_d}$

No net change in volume : At each pt in the fluid, the amt of fluid entering any infinitesimal volume is equal to the amt of fluid exiting that volume.

$v \cdot n = 0$ the component of the fluid's velocity that is directed outward through the boundary is 0. The fluid can't penetrate the boundary. (bed of a river)

Euler equation. acceleration of the fluid (both local and convective) is balanced by the pressure gradient.

local acceleration of the fluid.

how the velocity at a pt changes over time.

captures how the velocity of fluid particles changes as they move through the flow field.

"change in velocity due to the flow itself"

$$\begin{cases} \partial_t v^i + \sum_{j=1}^{d} v^j \partial_{x_j} v^i + \partial_{x_i} p = 0 & \text{in } \Omega \\ \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i} = 0 & \text{in } \Omega \\ v \cdot n = 0 & \text{on } \partial\Omega \end{cases}$$

$$\frac{d}{dt}\int_\Omega |v(t)|^2 = \frac{d}{dt}\int_\Omega \sum_{i=1}^{d} v^i(t)^2 = 2\int_\Omega \sum_{i=1}^{d} v^i \partial_t v^i$$

$$= 2\int_\Omega \sum_{i=1}^{d} v^i \left(-\sum_{j=1}^{d} v^j \partial_{x_j} v^i - \partial_{x_i} p\right)$$

$$= -2\int_\Omega \sum_{i,j=1}^{d} v^i v^j \partial_{x_j} v^i - 2\int_\Omega \sum_{i=1}^{d} v^i \partial_{x_i} p$$

$$= \underbrace{-\int_\Omega \sum_j v^j \partial_{x_j}\left(\sum_{i=1}^{d} (v^i)^2\right)}_{\downarrow \text{ by } \text{①}} - 2\underbrace{\int_\Omega \sum_{i=1}^{d} v^i \partial_{x_i} p}_{\downarrow \text{ by } \cdot p}$$

① ? (incomp) = Ⓐ + Ⓑ Ⓒ + Ⓓ.

$$\iiint_V (\nabla \cdot F)\, dV = \oiint_S (F \cdot \hat{n})\, dS$$

div thm

$$\int_{\partial\Omega} \langle v\|v\|^2, n\rangle = \int_\Omega \nabla \cdot (v\|v\|^2) = \int_\Omega \sum_j \partial_{x_j}(v^j \|v\|^2) = \int_\Omega \sum_j (\partial_{x_j} v^j)\|v\|^2 + \underline{\int_\Omega \sum_j v^j \partial_{x_j} \|v\|^2}$$

$g(t,x)$ describes the trajectory of a fluid particle starting from the initial position x, as it is advected by the velocity field $v(t,x)$ over $[0,T]$.

② ~~Why~~ $\cancel{\nabla_x g(t,x) = I_d}$?

Thm 2.3.15.

Euler eqts are equiv. to the geodesic eqt on $SDiff(\Omega) \subset L^2(\Omega;\mathbb{R}^d)$.

$SDiff(\Omega) := \{h: \Omega \to \Omega : h$ a measure preserving and orientation preserving diffeo$\}$.

Proof. Given $\bar{h} \in SDiff(\Omega)$, $t \mapsto h(t) \in S$ be a smooth curve of maps in $S(\Omega)$ with $h(0) = \bar{h}$, $\omega(t) := \partial_t h(t)$

$\Rightarrow \omega(t) \in T_{h(t)} SDiff(\Omega)$.

Since $h(t)$ is a diffeo of Ω, it maps $\partial\Omega$ onto itself. ?

$\omega(t) = \partial_t h(t)$ must be tangent to the boundary. Define $\tilde{\omega}(t) := \omega(t) \circ h^{-1}(t)$ so that $\tilde{\omega}(t)$ is also tangent to $\partial\Omega$.

$\partial_t h(t) = \tilde{\omega}(t, h(t))$.

Since $\det \nabla_x h(t,x) \equiv 1$, $0 = \frac{d}{dt} \det \nabla_x h(t,x) = \operatorname{div}(\tilde{\omega})(t, h(t,x)) \det \nabla_x h(t,x)$ $\to \operatorname{div}(\tilde{\omega}) = 0$.

Then taking $t=0$. $T_{\bar{h}} SDiff(\Omega) \subset \{\omega : \operatorname{div}(\omega \circ \bar{h}^{-1}) = 0,\ \omega \cdot \nu|_{\partial\Omega} = 0\}$.

$= \{\tilde{\omega} \circ \bar{h} \mid \operatorname{div}(\tilde{\omega}) = 0,\ \tilde{\omega} \cdot \nu|_{\partial\Omega} = 0\}$.

Given a vector field $\tilde{\omega}: \Omega \to \mathbb{R}^d$ with $\operatorname{div}(\tilde{\omega}) = 0$ and $\tilde{\omega} \cdot \nu|_{\partial\Omega} = 0$ we solve

$$\begin{cases} \partial_t h(t,x) = \tilde{\omega}(h(t,x)) \\ h(0,x) = \bar{h}(x). \end{cases}$$

Using the same computation. $\frac{d}{dt} \det \nabla h = 0$. Thus $h(t): \Omega \to \Omega$ is a curve in $SDiff(\Omega)$.

$\partial_t h(0) = \tilde{\omega} \circ \bar{h}$ is an element of the tangent space of $SDiff(\Omega)$ at $\bar{h}$.

$\Rightarrow \forall \bar{h} \in SDiff(\Omega)$. $T_{\bar{h}} SDiff(\Omega) = \{\tilde{\omega} \circ \bar{h} \mid \operatorname{div}(\tilde{\omega}) = 0,\ \tilde{\omega} \cdot \nu|_{\partial\Omega} = 0\}$.

Observe that (a) $\forall\ h_\# dx = dx$ map. $\langle f_1 \circ h, f_2 \circ h \rangle_{L^2} = \int_\Omega (f_1 \circ h)(x) \cdot (f_2 \circ h)(x)\, dx$

$= \int_\Omega f_1(x) \cdot f_2(x)\, dx$

$= \langle f_1, f_2 \rangle_{L^2}$.

(b). Every vector field in $L^2(\Omega, \mathbb{R}^d) := \{\omega : \Omega \to \mathbb{R}^d \mid \operatorname{div}(\omega) = 0 \quad \omega \cdot \nu|_{\partial\Omega} = 0\} \oplus \{\nabla g : g : \Omega \to \mathbb{R}\}$

the decomp. is orthogonal $\langle \omega, \nabla g \rangle_{L^2} = \int_\Omega \omega \cdot \nabla g\, dx$

$= -\int_{\partial\Omega} \omega \cdot \nu\, g - \int_\Omega \operatorname{div}(\omega) g\, dx = 0$.

To find the minimizing geodesics. consider $\inf\{\int_0^1\int_\Omega |\partial_t g(t,x)|^2 dx\,dt \mid g(t)\in SDiff,\ g(0)=g_0,\ g(1)=g_1\}$.

Given. $g_0, g_1 \in SDiff(\Omega)$. $\quad proj_{SDiff} : L^2(\Omega, \mathbb{R}^d) \to SDiff(\Omega)$. $\quad \frac{g_0+g_1}{2} \mapsto proj_{SDiff}\left(\frac{g_0+g_1}{2}\right)$.

Thm. Let $\Omega\subset\mathbb{R}^d$ be a bounded set with Lipschitz boundary, $d\geq 2$. Then $\overline{SDiff(\Omega)}^{L^2} = S(\Omega) = \{s:\Omega\to\Omega : s_\# dx = dx\}$

Thm. Let $h\in L^2(\Omega;\mathbb{R}^d)$ s.t. $h_\#(dx|_\Omega) \ll dx$. Then

(a) $\exists!$ ~~o~~ proj $\bar{s}$ onto $S(\Omega)$, it holds that $\|h-\bar{s}\|_{L^2(\Omega)} \leq \|h-s\|_{L^2(\Omega)}$ $\forall s\in S(\Omega)$.

(b) $\exists$ convex ψ s.t. $h=\nabla\psi\circ\bar{s}$.

Proof. (a) $h:\Omega\to\mathbb{R}^d$. $\quad \mu := h_\#(dx|_\Omega) \ll dx$. $\quad \int_{\mathbb{R}^d} d\mu = \int_{h(\Omega)} d\mu = \int_\Omega dx = |\Omega|$.

$\cdot e \quad dx(h^{-1}(B))$.

2.5.10. (Brenier) $X=Y=\mathbb{R}^d$ $\ c=\frac{|x-y|^2}{2}$ or $c(x,y)=-xy$. Suppose $\int_{\mathbb{R}^d}|x|^2 d\mu + \int_{\mathbb{R}^d}|y|^2 d\nu < \infty$ and $\mu \ll dx$.
Then $\exists!$ optimal plan $\bar{\gamma}$. $\bar{\gamma} = (Id\times T)_\#\mu$ and $T=\nabla\varphi$ for some convex φ.

Cor. 2.5.13. Assume Brenier, $\nu\ll dx$. $\nabla\varphi$ be the optimal transport map from μ to ν, and $\nabla\psi$ be the optimal transport map from ν to μ. Then $\nabla\varphi\circ\nabla\psi = Id$ ν a.e. and $\nabla\psi\circ\nabla\varphi = Id$ μ a.e.

Thus, $\exists$ convex $\varphi,\psi:\mathbb{R}^d\to\mathbb{R}$ s.t. $\nabla\varphi$ and $\nabla\psi$ are optimal from μ to $dx|_\Omega$ and vice versa.

Let $\bar{s} := \nabla\varphi\circ h : \Omega\to\Omega$.

$$\begin{aligned}\int_\Omega |\bar{s}(x) - h(x)|^2 dx &= \int_\Omega |\nabla\varphi\circ h - h|^2 dx \\ &= \int_{\mathbb{R}^d} |\nabla\varphi - Id|^2 d\mu \\ &= \min_{\gamma\in\Gamma(\mu, dx|_\Omega)} \int_{\mathbb{R}^d\times\Omega} |x-y|^2 d\gamma.\end{aligned}$$

If $s\in S(\Omega)$, then $\gamma_s := (h\times s)_\#(dx|_\Omega) \in \Gamma(\mu, dx|_\Omega)$.

$(\pi_x)_\#\gamma_s = h_\#(dx|_\Omega) = \mu$ and $(\pi_y)_\#\gamma_s = s_\#(dx|_\Omega) = dx|_\Omega$.

$$\Rightarrow \min_{\gamma\in\Gamma(\mu, dx|_\Omega)} \int_{\mathbb{R}^d\times\Omega} |x-y|^2 d\gamma \leq \min_{s\in S(\Omega)} \int_{\mathbb{R}^d\times\Omega} |x-y|^2 d\gamma_s = \min_{s\in S(\Omega)} \int_\Omega |h(x)-s(x)|^2 dx.$$

$$\Rightarrow \int_\Omega |h(x)-\bar{s}(x)|^2 dx \leq \min_{s\in S(\Omega)} \int_\Omega |h(x)-s(x)|^2 dx.$$ Thus $\bar{s}$ is a projection.

Suppose $\hat{s}$ is another projection. $\gamma_{\bar{s}}, \gamma_{\hat{s}}$ are both optimal couplings. Thus, by 2.5.10 ($\exists!$)

$$\int_\Omega F(h(x), \hat{s}(x))\,dx = \int_\Omega F(h(x), \bar{s}(x))\,dx \quad \forall F\in C_b(\mathbb{R}^d\times\mathbb{R}^d)$$

Choosing $F(x,y) = |\nabla\varphi(x) - y|^2$ and recall $\bar{s} = \nabla\varphi\circ h$.

$$0 = \int_\Omega |\nabla\varphi\circ h - \hat{s}|^2 dx = \int_\Omega |\bar{s}-\hat{s}|^2 dx. \qquad \hat{s} = \bar{s}.$$

(b) follows from $\bar{s} = \nabla\varphi\circ h$ $\quad \nabla\psi = (\nabla\varphi)^{-1}$

Remark: (a). $\forall M \in \mathbb{R}^{d\times d}$ $M = SO$. S symmetric ~~psd~~ psd. O orthogonal.

$h(x) = Mx$. $h = \nabla\varphi \circ \bar{s}$ $\varphi(x) = \frac{1}{2}\langle x, Sx\rangle$ $\bar{s}(x) = Ox$.

(b). Consider a smooth vector field $\omega: \mathbb{R}^d \to \mathbb{R}^d$. let $h_t(x) := h(t,x)$ be the flow of ω.

$$\begin{cases} \partial_t h(t,x) = \omega(h(t,x)) \\ h(0,x) = x. \end{cases}$$

Then $h_\varepsilon(x) = h_0(x) + \partial_t h_t(x)\big|_{t=0}\, \varepsilon + o(\varepsilon)$

$= x + \varepsilon\omega(x) + o(\varepsilon)$.

The PD of h_ε yields $h_\varepsilon = \nabla\psi_\varepsilon \circ s_\varepsilon$

We suppose that $\psi_\varepsilon(x) = \frac{|x|^2}{2} + \varepsilon g(x) + o(\varepsilon)$ $s_\varepsilon(x) = x + \varepsilon u(x) + o(\varepsilon)$.

Also since $\det \nabla s_\varepsilon = \det(Id + \varepsilon\nabla u + o(\varepsilon)) = 1 + \varepsilon\, \mathrm{div}(u) + o(\varepsilon)$.

and s_ε is measure preserving (hence $1 \equiv \det \nabla s_\varepsilon$), $\mathrm{div}(u) = 0$.

Hence $x + \varepsilon\omega(x) + o(\varepsilon) = h_\varepsilon = \nabla\psi_\varepsilon \circ s_\varepsilon$.

$= (x + \varepsilon\nabla g(x)) \circ (x + \varepsilon u(x)) + o(\varepsilon)$.

$= x + \varepsilon(u(x) + \nabla g(x)) + o(\varepsilon)$.

$\Rightarrow$. $\omega = u + \nabla g$.

Any vector field ω can be written as the sum of a div free vector field and a gradient. which is the Helmholtz decomp.

$\Rightarrow$ Helm decomp. is the infinitesimal ver. of the polar decomp.