$C_c^\infty(U)$ denote the space of infinitely diffble functions $\phi: U \to \mathbb{R}$, with cpt support in U.

We call $\phi \in C_c^\infty(U)$ a test function. Intuition: From by part $\int_U \underset{\text{what if not diffble}}{u_{x_i}}' \phi \, dx = -\int_U u \phi_{x_i} dx$ — weak derivative; always make sense

Definition 3.5. A sequence $\phi_j \in C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$ if there exists a compact set $K \subset U$ such that $\operatorname{supp} \phi_j, \operatorname{supp} \phi \subseteq K$ and

$$\lim_{j \to \infty} \sup_{x \in K} |D^\alpha \phi_j(x) - D^\alpha \phi(x)| = 0$$

for every multi-index α.

Some references I've read:
- Sung jin's 222 AB lecture notes
- Brezis Functional Analysis
- Evans PDE
- Hörmander The analysis of linear PDE vol 1.

Definition 3.7. A *distribution* u on U is a linear functional $u : C_c^\infty(U) \to \mathbb{R}$ that is *continuous* in the following sense: For any sequence $\phi_j \in C_c^\infty(U)$ such that $\phi_j \to \phi \in C_c^\infty(U)$ in the sense of Definition 3.5, then $u(\phi_j) \to u(\phi)$.

$u \in C_c^\infty(U)^*$, we often use duality pairing $\langle u, \phi \rangle := u(\phi)$

Eg. 1 Any signed Borel measure μ defined a distribution by $\langle \mu, \phi \rangle = \int \phi(x) \, d\mu(x)$

Eg. 2. $u: U \to \mathbb{R}$ is locally integrable if it's msble and abs. integrable on every cpt subset $K \subseteq U$ w.r.t. the Lebesgue measure $\int_K |f| < \infty$.

$L^1_{loc}(U)$ is the space. $\Rightarrow \langle u, \phi \rangle = \int u \phi \, dx$

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0 \qquad \text{in } \mathbb{R}^d \times (0, T),$$

$\partial_t \mu_t$ does not directly mean taking the derivative of measure, but in the sense of distributions!

$\rightarrow$ Given $u \in \mathcal{D}'(U)$, $\langle \partial_j u, \phi \rangle := -\langle u, \partial_j \phi \rangle$ $\forall \phi \in C_c^\infty(U)$ (by part)

Among all the OT books, Villani's Chp 8 is the easiest, he assumes μ_t is the density of a measure when giving intuition.

But everything should be in weak sense in FA / PDE

perhaps identify it using Riesz's

$$\int_0^T \int_{\mathbb{R}^d} (\partial_t \mu_t + \nabla \cdot (v_t \mu_t)) \varphi(x,t)\, dx\, dt = 0$$

$$\langle \nabla \cdot (v_t \mu_t), \phi \rangle = -\langle v_t \mu_t, \nabla \phi \rangle$$

(that's why I understand nothing 2 week's ago)

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \mu_t \, \varphi(x,t)\, dx\, dt + \int_0^T \int_{\mathbb{R}^d} \nabla \cdot (v_t \mu_t) \varphi(x,t)\, dx\, dt = 0$$

$$-\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(x,t)\, d\mu_t(x)\, dt - \int_0^T \int_{\mathbb{R}^d} \langle v_t(x), \nabla_x \varphi(x,t) \rangle\, d\mu_t(x)\, dt = 0$$

$$\int_\Gamma u V \cdot \hat{n}\, d\Gamma = \int_\Omega \nabla \cdot (uV)\, d\Omega = \int_\Omega u \nabla \cdot V\, d\Omega + \int_\Omega \nabla u \cdot V\, d\Omega.$$ Divergence Thm.

$$\int_0^T \int_{\mathbb{R}^d} \partial_t \varphi(x,t)\, d\mu_t(x)\, dt + \int_0^T \int_{\mathbb{R}^d} \langle \nabla_x \varphi(x,t), v_t(x) \rangle\, d\mu_t(x)\, dt = 0$$

If we think about $\begin{cases} \dot{X}(t,x) = v(t, X(t,x)) \\ X(0,x) = x \end{cases}$ $\quad \mu_t = (X(t))_\# \mu_0$

$$\int_{\mathbb{R}^d} \partial_t \mu_t(x) \varphi(x)\, dx = \frac{d}{dt} \int_{\mathbb{R}^d} \mu_t(x) \varphi(x)\, dx = \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(X(t,x)) \mu_0(x)\, dx$$

$$= \int_{\mathbb{R}^d} \nabla \varphi(X(t,x)) \cdot \dot{X}(t,x) \mu_0(x)\, dx = \int_\Omega \nabla \varphi(X(t,x)) \cdot v_t(X(t,x)) \mu_0(x)\, dx$$

$$= \int_\Omega \nabla \varphi(x) \cdot v_t(x) \mu_t(x)\, dx = -\int_\Omega \varphi(x)\, \mathrm{div}(v_t \mu_t)\, dx.$$

Why continuity eqt holds in the representation of Prop. 8.18

Banach's fixed pt thm.

Def^{n} (Contraction) Let (X,d) be a complete metric space. $T: X \to X$ is called a contraction if $0<\theta<1$,

$$d(Tx,Ty) \le \theta(x,y) \quad \forall x,y \in X$$

Any contraction T must possess a unique fixed point in X. $(Tx^* = x^*)$

Proof: 1) Uniqueness $Tx=x$, $Ty=y$

$$d(x,y) = d(Tx,Ty) \le \theta\, d(x,y) \Rightarrow d(x,y)=0$$

2) Existence take any $x_0 \in X$. $x_{n+1} = Tx_n$ $n \ge 0$

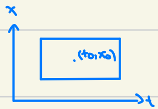
We show that (x_n) is a Cauchy sequence: $n > m \ge N$

$$d(x_n,x_m) \le d(x_n,x_{n-1}) + \cdots + d(x_{m+1},x_m) \le (\theta^{n-1} + \cdots + \theta^{m})\, d(x_1,x_0).$$

$$d(x_n,x_{n-1}) = d(Tx_{n-1}, Tx_{n-2}) \le \theta\, d(x_{n-1},x_{n-2}) \le \cdots \le \theta^{n-1} d(x_1,x_0)$$

Picard's thm for ODE:

$$\begin{cases} \frac{d}{dt}x(t) = f(t,x(t)) \\ x(t_0) = x_0 \end{cases}$$



Assume f is Lipschitz in the rectangle $R = \{|t-t_0| \le a,\ |x-x_0| \le b\}$

$$|f(t,\tilde{x}) - f(t,x)| \le k|\tilde{x}-x| \quad \forall \tilde{x},x \in R. \quad k>0.$$

$$\|f\|_\infty \le M$$

For some $\beta>0$, the ODE (*) has a unique C^1 sol $x(t)$ for $|t-t_0| \le \beta$.

Proof: (Apply the "spirit" of fixed pt thm)

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds$$

It suffices for us to prove existence and uniqueness in $C^0([t_0-\beta, t_0+\beta])$ $= X$

Given $x \in X$ $\quad (Tx)(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds$

1) T maps X to X ($x(s)$ is cts $\Rightarrow$ $f(s,x(s))$ is cts)

2) T is a contraction: take any $x(t), y(t) \in X$

$$|(Tx)(t) - (Ty)(t)| \le \left|\int_{t_0}^{t} |f(s,x(s)) - f(s,y(s))|\,ds\right|$$

what if $t<t_0$

$$\le \left|\int_{t_0}^{t} k|x(s)-y(s)|\,ds\right| \le k|t-t_0| \max_{|s-t_0|\le\beta} |x(s)-y(s)|$$

$$\|Tx-Ty\|_X \le k\beta \|x-y\|_X$$ need to take $k\beta<1$.