look for all the paths μ_t in $\mathcal{M}_+(X)$ connecting μ_0, μ_1. Ω open $\subseteq \mathbb{R}$

Suppose $\mu_t = \delta_{x(t)}$, it turns out that the velocity field $E_t = \dot{x}(t)\delta_{x(t)}$ is related to μ_t by the equation $\dot{\mu}_t + \nabla E_t = 0$ in $(0,1)\times\Omega$ in the distribution sense.

Given $\varphi \in C_c^\infty(0,1)$, $\phi \in C_c^\infty(\Omega)$, we can take $\varphi(t)\phi(x)$ as test function

$$(\dot{\mu}_t + \nabla\cdot E_t)(\varphi\phi) = -\int_0^1 \dot{\varphi}(t)\phi(x(t)) + \varphi(t)\langle \nabla\phi(x(t)), \dot{x}(t)\rangle dt$$
$$= -\int_0^1 \frac{d}{dt}[\varphi(t)\phi(x(t))]dt = 0$$

The eq^n holds iff $\dot{\mu}_t(\phi) = \nabla\phi\, E_t$ in $(0,1)$ $\forall \phi \in C_c^\infty(\Omega)$

We can test the first equation $\langle \dot{\mu}_t, \varphi\rangle + \langle \nabla\cdot E_t, \varphi\rangle = 0$

IBP : $\langle \dot{\mu}_t, \varphi\rangle = \langle E_t, \nabla\varphi\rangle$

Recall that $f:(0,1)\to(E,d)$ is abs. cts if $\forall \varepsilon>0, \exists \delta>0$ s.t.
$\sum_i (y_i - x_i) < \delta \Rightarrow \sum_i d(f(y_i), f(x_i)) < \varepsilon$ $\forall$ partition of $(0,1)$

Prop. If some family $(\mu_t) \subset \mathcal{M}_1(X)$ satisfy the continuity equation for suitable measures $E_t \in [M(\Omega)]^n$ s.t. $\int_0^1 |E_t|(\Omega)dt < \infty$ then f is an abs. cts map between $(0,1)$ and $\mathcal{M}_1(X)$, endowed with the 1-Wasserstein distance

$$\lim_{h\to 0} \frac{W_1(\mu_{t+h}, \mu_t)}{|h|} \le |E_t|(\Omega) \quad \text{for } \mathcal{L}^1 \text{ a.e. } t\in(0,1)$$

Let $\mu_0, \mu_1 \in \mathcal{M}_1(X)$ be given prob measures Minimize $J(E) := \int_0^1 |E_t|(\Omega)dt$ among all Borel maps $\mu_t : [0,1] \to \mathcal{M}_+(X)$ and $E_t : [0,1] \to [M(\Omega)]^n$ s.t. the continuity equation holds. This is the OT problem by Brenier.

E.g. $\mu_t = 1_{[t,t+1]}\lambda$ $\quad E_t = 1_{[t,t+1]}\lambda$ (Lebesgue measure) The OT map is $x \mapsto x+1$.

Consider flow
$$\mu_t = \begin{cases} \frac{1}{1-2t} 1_{[2t,1]}\lambda & 0\le t<\frac{1}{2} \\ \delta_1 & t=\frac{1}{2} \\ \frac{1}{2t-1} 1_{[1,2t]}\lambda & \frac{1}{2}<t\le 1 \end{cases}$$

$$E_t = \begin{cases} \frac{2(1-x)}{(1-2t)^2} 1_{[2t,1]}\lambda & 0\le t<\frac{1}{2} \\ \frac{2(x-1)}{(2t-1)^2} 1_{[1,2t]}\lambda & \frac{1}{2}<t\le 1 \end{cases}$$

It's ez to check that $\dot{\mu}_t - \nabla\cdot E_t = 0$

$|E_t|$ are prob. measures $\forall t \neq \frac{1}{2}$, hence $J(E) = 1$.

For the ODE $\dot{f}_t + \nabla(g_t f_t) = 0$. If f_t, g_t are smooth w.r.t. the space and time variables, uniqueness is a consequence of the classical method of characteristics,

$$f_t(x_t) = f_0(x) \exp\left(-\int_0^t a(x_s) ds\right)$$

$c_t = \nabla \cdot g_t$ and x_t solves the ODE $\dot{x}_t = g_t(x_t)$, $x_0 = x$ $t \in (0,1)$

Thm The ODE problem has at least 1 solution and $\min(\text{ODE}) = \min(\text{MK})$.

$\forall$ optimal coupling γ for (MK) the measures

$$\mu_t := \pi_t \# \gamma \qquad E_t := \pi_t \# ((y-x)\gamma) \qquad t \in [0,1]$$

with $\pi_t(x,t) = x + t(y-x)$ solve (ODE).